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# Summing one-loop graphs at multiparticle threshold

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It is shown that the technique recently suggested by Brown for summing the tree graphs at threshold can be extended to calculate the loop effects. An explicit result is derived for the sum of one-loop graphs for the amplitude of threshold production of $n$ on-mass-shell particles by one virtual particle in the unbroken $\lambda\phi^4$ theory. It is also found that the tree-level amplitude of production of $n$ particles by two incoming on-mass-shell particles vanishes at the threshold for $n > 4$.

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The problem of calculating amplitudes of processes with many weakly interacting particles has recently attracted a considerable interest, initially triggered by the observation [1–3] that such processes in particular are associated with a possible baryon- and lepton-number violation in high-energy electroweak interactions. Cornwall [4] and Goldberg [5] have pointed out that in perturbative amplitudes with many external particles the weak coupling may get compensated by a large number of diagrams. This is a manifestation of the old-standing problem of the factorial growth of the coefficients in the perturbation theory [6]. Since the perturbative expansion for multiparticle amplitudes starts from a high order in the coupling constant, for a sufficiently large number $n$ of particles the factorial growth of the coefficients in the series invalidates the perturbative calculation of such amplitudes. Given the lack of a better approach it seems useful to quantify and study the problem within the perturbation theory itself. A simple model example in which the problem arises with full strength is the amplitude $A_n$ where a virtual particle of a real scalar field $\phi$ produces a large number $n$ of on-mass-shell $\phi$ particles in the $\lambda\phi^4$ theory. It has been recently found that the sum of all tree graphs for this amplitude in the threshold limit, i.e., when all the produced particles are at rest, can be calculated exactly for arbitrary $n$ both in the case of unbroken symmetry [7] and in the case of theory with spontaneous breaking of the symmetry under the reflection $\phi \rightarrow -\phi$ [8].

Originally the calculation [7] was done by directly solving a recursion relation for the tree graphs. Argyres, Kleiss, and Papadopoulous [8] applied a regular method of solving the recursion relations based on a generating function for which the recursion relation for the amplitudes $A_n$ is equivalent to second-order nonlinear differential equation. Most recently Brown [9] has shown that the generating function is nothing else than the classical field $\phi_0(t)$ generated by an external source $\rho=\rho_0 e^{imt}$ which field is a complex solution of the Euler-Lagrange classical equation satisfying the condition that it has only the positive frequency part. The equation and their solutions in both approaches are related by a simple change of variable. Thus Brown has reproduced the previous results [7,8] in a simple and elegant way.

The purpose of this paper is to show that Brown's technique can be extended to calculate the loop contributions to the amplitudes $A_n$ as well as the amplitudes of more complicated processes, e.g., of the scattering $2 \rightarrow n$. For definiteness the case of unbroken $\lambda\phi^4$ theory with the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{2}m^2\phi^2-\tfrac{1}{4}\lambda\phi^4 \tag{1}$$

will be considered and we will concentrate on the calculation of the one-loop correction to the threshold amplitudes $A_n$. The result of this calculation can be written as

$$\langle 2k+1|\phi(0)|0\rangle=(2k+1)!\left[\frac{\bar{\lambda}}{8\bar{m}^2}\right]^k \times\left[1-k(k-1)\frac{3^{3/2}\lambda}{16\pi^2} \times\left[\ln\frac{2+\sqrt{3}}{2-\sqrt{3}}-i\pi\right]\right], \tag{2}$$

where $\bar{\lambda}$ and $\bar{m}$ are the renormalized coupling constant and the mass; the appropriate renormalization condition, which specifies the finite terms, will be described below. Equation (2) gives the exact sum of all tree and one-loop graphs at the threshold of production of $n=2k+1$ $\phi$ particles for arbitrary $k$.[1] The formula (2) gives the relative magnitude of the first loop correction growing as $n^2$ at large $n$. This behavior explicitly demonstrates invalidity of the previous arguments [10,11] of the present author that terms containing $n^2\lambda$ should be absent in the loop effects. The reason for those arguments to be faulty is related to the singularities of the underlying classical field in the plane of complex time, where the quantum fluctuations are more singular than the classical background. However, the presence of the $n^2\lambda$ parameter in the loop effects does not necessarily imply that the quantum effects completely eliminate the growth of the amplitudes and a further study is needed. The relatively easy calculability of the threshold amplitudes at the tree and one-loop levels and the remarkably simple form of the result

[1]The number of $n$ of final particles produced by one virtual is necessarily odd due to the unbroken reflection symmetry.

 

(2) gives us a hope that these amplitudes can be studied well beyond the first terms in the perturbative expansion.

That the amplitudes in the $\lambda\phi^4$ theory at the multiparticle thresholds may have special properties is also hinted at by another fact, which follows as a by-product from the calculation in this paper. Namely, if one considers the sum of all tree graphs for the amplitude of the process where two incoming on-shell particles produce $n$ on-shell particles exactly at the threshold, it turns out that this sum is nonzero only for $n=2$ and $n=4$ and is vanishing for all $n>4$ (the number of final particles in this case is necessarily even). It is in fact due to this behavior that the one-loop term in Eq. (2) contains only the four-particle factor.

The technique suggested by Brown [9] is based on the standard reduction formula representation of the amplitude through the response of the system to an external source $\rho(x)$, which enters the term $\rho\phi$ added to the Lagrangian:

$$\langle n|\phi(x)|0\rangle = \left[\prod_{a=1}^{n} \lim_{p_a^2\to m^2} \int d^4x_a e^{ip_a x_a}(m^2-p_a^2)\frac{\partial}{\partial\rho(x_a)}\right]\langle 0_{\rm out}|\phi(x)|0_{\rm in}\rangle^{\rho}\Big|_{\rho=0} , \quad (3)$$

the tree-level amplitude being generated by the response in the classical approximation, i.e., by the classical solution $\phi_0(x)$ of the field equations in the presence of the source.

For all the spatial momenta of the final particles equal to zero it is sufficient to consider the response to a spatially uniform time-dependent source $\rho(t)=\rho_0(\omega)e^{i\omega t}$ and take the on-mass-shell limit in Eq. (3) by tending $\omega$ to $m$. The spatial integrals in Eq. (3) then give the usual factors with the normalization spatial volume, which as usual is set to one, while the time dependence on one common frequency $\omega$ implies that the propagator factors and the functional derivatives enter in the combination

$$(m^2-p_a^2)\frac{\partial}{\partial\rho(x_a)}\to(m^2-\omega^2)\frac{\partial}{\partial\rho(t)}=\frac{\partial}{\partial z(t)} , \quad (4)$$

where

$$z(t)=\frac{\rho_0(\omega)e^{i\omega t}}{m^2-i\epsilon-\omega^2} \quad (5)$$

coincides with the response of the field to the external source in the limit of absence of the interaction, i.e., of $\lambda=0$. For a finite amplitude $\rho_0$ of the source the response $z(t)$ is singular in the limit $\omega\to m$. The crucial observation of Brown [9] is that, since according to Eq. (4) we need the dependence of the response of the interacting field $\phi$ only in terms of $z(t)$, one can take the limit $\rho_0(\omega)\to 0$ simultaneously with $\omega\to m$ in such a way that $z(t)$ is finite:

$$z(t)\to z_0 e^{imt} . \quad (6)$$

Furthermore, to find the classical solution $\phi_0(x)$ in this limit one does not have to go through this limiting procedure, but rather consider directly the on-shell limit with vanishing source. The field equation with zero source is of course given by

$$\partial^2\phi+m^2\phi+\lambda\phi^3=0 . \quad (7)$$

For the purpose of calculating the matrix element in Eq. (3) at the threshold one looks for a solution of this equation which depends only on time and contains only the positive frequency part with all harmonics being multiples of $e^{imt}$. The solution satisfying these conditions reads as [9]

$$\phi_0(t)=\frac{z(t)}{1-(\lambda/8m^2)z(t)^2} . \quad (8)$$

According to Eqs. (4) and (3) the $n$th derivative of this solution with respect to $z$ gives the matrix element $\langle n|\phi(0)|0\rangle$ at the threshold in the tree approximation:

$$\langle 2k+1|\phi(0)|0\rangle_0=\left[\frac{\partial}{\partial z}\right]^{2k+1}\phi_0\Big|_{z=0} =(2k+1)!\left[\frac{\lambda}{8m^2}\right]^k , \quad (9)$$

which reproduces the previously known result [7]. The fact that the matrix element is nonzero only for odd $n$ obviously follows from that the expansion of $\phi_0$ in Eq. (8) contains only odd powers of $z$.

It can be noticed that the solution (8) is in fact not uniquely determined by the above-mentioned conditions. Namely, $z(t)$ can be rescaled by an arbitrary constant $C$. This constant corresponds to the choice of normalization of the field, so that the value of $C=1$ is fixed by the usual normalization condition $\langle 1|\phi(0)|0\rangle=1$, as can be seen from the linear term in the expansion of $\phi_0$ in powers of $z$.

Another important point concerning the solution (8) is related to the fact that this solution is essentially complex for real time $t$. This is imposed by the fact that in calculating production of particles by the virtual field, rather than both production and absorption, one necessarily has to consider only the positive frequency part of the field, which is essentially complex.

The quantum loop corrections to the amplitudes $\langle n|\phi|0\rangle$ are obtained by substituting instead of the classical field the mean value of the full field,

$$\phi(x)=\phi_0(x)+\phi_q(x) , \quad (10)$$

where $\phi_q(x)$ is the quantum part of the field. Expanding the field equation (7) near the classical solution $\phi_0$ and retaining only the first nonvanishing quantum correction, one finds that the mean field $\phi(x)$ to the first quantum order satisfies the equation

$$\partial^2\phi(x)+m^2\phi(x)+\lambda\phi(x)^3+3\lambda\phi_0(x)\langle\phi_q(x)\phi_q(x)\rangle=0\ , \tag{11}$$

where $\langle\phi_q(x)\phi_q(x)\rangle$ is the limit of the Green's function in the classical background field $\phi_0$,

$$G(x_1,x_2)=\langle T(\phi_q(x_1)\phi_q(x_2))\rangle\ , \tag{12}$$

when its arguments are at the same point $x$.

Therefore the steps needed to calculate the first loop correction to the amplitudes $A_n$ are the following: (i) Calculate the Green's function (12) as the inverse of the operator of the second variation of the action:

$$\partial^2+m^2+3\lambda\phi_0(x)^2\ ; \tag{13}$$

(ii) find its limit in coinciding points, which enters Eq. (11); (iii) expand the solution of thus found equation in powers of $z(t)$, which gives the amplitudes at the threshold in the same way as in Eq. (9).

This program, however, is obscured at the very first step by the fact that with essentially complex $\phi_0$ [Eq. (8)] the operator (13) is essentially non-Hermitian. However, one can render this operator real and thus the problem more tractable by analytical continuation in time $t$, which amounts to rotation and shift in the complex plane. Namely, the substitution which achieves the goal reads as

$$\left[\frac{\lambda}{8m^2}\right]^{1/2} z(t)=ie^{m\tau} \tag{14}$$

and the variable

$$\tau=it+\frac{1}{2m}\ln\frac{\lambda z_0^2}{8m^2}-\frac{i\pi}{2m} \tag{15}$$

is then used as the new time variable $t$. The necessity of the shift in addition to the usual rotation to the Euclidean time is caused by existence of a pole of $\phi_0(t)$ on the negative imaginary axis, where the operator (13) is singular. The poles are repeated parallel to the real axis with the period $\pi/m$. The axis, corresponding to real $\tau$, on which the operator (13) is real runs parallel to the imaginary axis of $t$ exactly in the middle between two poles; see Fig. 1. It should be emphasized, however, that it is the pole

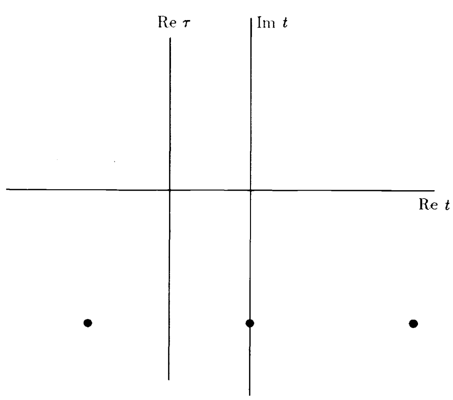


FIG. 1. The structure of the classical field $\phi_0(t)$ [Eq. (8)] in the complex $t$ plane. Heavy dots indicate the poles of $\phi_0$. The vertical line going between the poles is the axis of real $\tau$ on which the operator (13) is real.

structure of the field, which gives rise to the factorial growth of the multiparticle amplitudes the study of which may eventually be the central point in solving the problem of multiboson processes. Here, for the purpose of the specific calculation, we chose to stay away from the poles to avoid explicit singularity in the equations.

To somewhat simplify the notation we set the mass $m$ equal to one and restore it when needed and also introduce the notation $u(\tau)=e^{\tau}=-i\sqrt{\lambda/8}z(t)$. For real $u(\tau)$ the classical field (8) is purely imaginary:

$$\phi_0[u(\tau)]=\left[\frac{8}{\lambda}\right]^{1/2}\frac{iu}{1+u^2}=\left[\frac{2}{\lambda}\right]^{1/2}\frac{i}{\cosh\tau} \tag{16}$$

and the operator (13) is real. In a mode with spatial momentum $\mathbf{k}$ the operator has the form

$$-\frac{d^2}{d\tau^2}+\omega^2-\frac{6}{(\cosh\tau)^2}\ , \tag{17}$$

which is the familiar operator in one of exactly solvable potentials in quantum mechanics (see, e.g., Ref. [12]), and $\omega$ is the energy of the mode: $\omega^2=\mathbf{k}^2+1$.

The regular at $\tau\to+\infty$ solution of the homogeneous equation with the operator (17) has the form

$$f_1[u(\tau)]=\frac{2-3\omega+\omega^2-8u^2+2\omega^2u^2+2u^4+3\omega u^4+\omega^2u^4}{u^{\omega}(1+u^2)^2} \tag{18}$$

and the solution regular at $\tau\to-\infty$ is given by

$$f_2[u(\tau)]=f_1[1/u(\tau)]=\frac{u^{\omega}(2+3\omega+\omega^2-8u^2+2\omega^2u^2+2u^4-3\omega u^4+\omega^2u^4)}{(1+u^2)^2}\ . \tag{19}$$

The Wronskian of these solutions is given by

$$W=f_1(\tau)f_2'(\tau)-f_1'(\tau)f_2(\tau)=2\omega(\omega^2-1)(\omega^2-4)\ . \tag{20}$$

The convention for the sign of the Green's function used here is specified by the explicit expression for the Green's function in partial wave with the spatial momentum $\mathbf{k}$ in terms of $f_1$, $f_2$, and $W$: $G_\omega(\tau_1,\tau_2)=f_1(\tau_1)f_2(\tau_2)/W$ for $\tau_1>\tau_2$ and $G_\omega(\tau_1,\tau_2)=f_1(\tau_2)f_2(\tau_1)/W$ for $\tau_2>\tau_1$.

Naturally, having the explicit expression for the Green's function one can also evaluate amplitudes of more complicated processes, say, the tree-level amplitude of the threshold production of $n$ particles by two incoming on-mass-shell particles of high energy. However, Eqs. (18)–(20) show that there is in fact almost nothing to calculate for the latter amplitude: the Green's function has poles only at $\omega^2=1$ and $\omega^2=4$ [the zeros of the Wronskian (20)]. By the reduction formula this implies that the on-mass-shell amplitude is nonvanishing only at these values of the energy of each of the two incoming particles. The case $\omega=1$ corresponds to the trivial process $2\to2$ at the threshold, while the case $\omega=2$ corresponds to the process $2\to4$. (In the rest frame of the produced particles, which is used throughout this paper, $\omega$ corresponds to the energy of each of the two incoming particles, so that the total energy is $2\omega=4$.) The absence of other poles of the Green's function at higher $\omega$ means that for $n>4$ the sum of tree graphs for the on-shell process $2\to n$ is vanishing.

After this remark we proceed with calculating the loop correction in Eq. (2). The partial wave Green's function at coinciding points is given by

$$g_\omega(\tau)=G_\omega(\tau,\tau)=f_1(\tau)f_2(\tau)/W\ , \tag{21}$$

which yields the average value of the square of quantum fluctuations after integrating over $\mathbf{k}$:

$$\langle\phi_q(\tau)\phi_q(\tau)\rangle=g(\tau)=\int g_\omega(\tau)\frac{d^3k}{(2\pi)^3}$$
$$=\frac{1}{2\pi^2}\int_1 g_\omega(\tau)\omega\sqrt{\omega^2-1}\,d\omega\ . \tag{22}$$

The calculation of the latter integral involves problems related to the on-shell singularities and to the ultraviolet divergence. The on-shell singularities correspond to the zeros of the Wronskian (20) at $\omega^2=1$ and $\omega^2=4$. The first of these corresponds to the translational zero mode of the classical solution $\phi_0$ and in fact produces no effect in the integral in Eq. (22) since the singularity at $\omega^2=1$ is integrable. (This is why in the four-dimensional theory one does not have to consider subtraction of the contribution of the zero mode from the Green's function.) The pole at $\omega^2=4m^2$ (the dependence on mass is restored) is dealt with using Feynman's $i\epsilon$ rule, i.e., by shifting the pole to the negative half-plane $m^2\to m^2-i\epsilon$. The integral then develops imaginary part, which in the end corresponds to the dynamical imaginary part of the one-loop graphs, dictated by the unitarity.

To separate the ultraviolet divergent terms we expand $g_\omega(\tau)$ in powers of $\omega^{-1}$ and find that the two terms, which give the quadratic and the logarithmic divergence, have the form

$$g_\omega(\tau)=\frac{1}{2\omega}+\frac{6u^2}{(1+u^2)^2\omega^3}+g^r_\omega(\tau)\ , \tag{23}$$

where the regular part $g^r_\omega(\tau)$ contains terms of the order $\omega^{-5}$ and higher, so that its contribution to the integral in Eq. (22) is finite in the ultraviolet. After this decomposition the result of the integration in Eq. (22) can be presented as

$$g(\tau)=\frac{1}{2}I_1+\frac{6u^2}{(1+u^2)^2}\left[I_2+\frac{1}{2\pi^2}\right]-\frac{6u^4}{(1+u^2)^4}F\ , \tag{24}$$

where

$$F=\frac{\sqrt{3}}{2\pi^2}\left[\ln\frac{2+\sqrt{3}}{2-\sqrt{3}}-i\pi\right] \tag{25}$$

and $I_1$, $I_3$ are the ultraviolet divergent integrals:

$$I_n=\frac{1}{2\pi^2}\int\omega^{1-n}\sqrt{\omega^2-1}\,d\omega\ . \tag{26}$$

In Eq. (24) we have also combined with the logarithmically divergent integral a part of the finite contribution from integration of the $g^r_\omega(\tau)$, which has the same functional dependence on $u(\tau)$, hence the factor $[I_3+1/(2\pi^2)]$.

The divergent contributions can be regularized in a standard way, the most straightforward being the Pauli-Villars regularization. Upon substitution into Eq. (11) for the mean field with the quantum correction the quadratically divergent part proportional to $I_1$ gives rise to a term linear in the classical field $\phi_0$ while the logarithmically divergent part proportional to $I_3$ results in a correction to the term with $\lambda\phi_0^3$. Therefore these terms can be dumped into the definition of the renormalized mass $\overline{m}$ and the coupling constant $\overline{\lambda}$ according to

$$\overline{m}^2=m^2+\frac{3\lambda}{2}I_1\ ,$$
$$\overline{\lambda}=\lambda-\frac{9\lambda^2}{4}\left[I_3+\frac{1}{2\pi^2}\right]\ . \tag{27}$$

These definitions can be used to relate the quantities $\overline{m}$ and $\overline{\lambda}$ to the renormalized constants in any other renormalization scheme. One can readily see that the divergent parts are scheme independent, while the relation between the finite parts depends on the specific definition of the regularization procedure.

The only nontrivial modification of the average field given by Eq. (11) is related to the finite part of the average value of the square of quantum fluctuations [Eq. (22)], proportional to the constant factor $F$. If one seeks the solution of the equation (11) in the form $\phi(t)=\phi_0(t;\overline{m},\overline{\lambda})+\phi_1(t)$, where the renormalization of the constants is plugged into the functional dependence of the classical solution, the equation for the correction $\phi_1(\tau)$ (i.e., on the $\tau$ axis) reads as

$$\left[\frac{d^2}{d\tau^2}-1+\frac{24u^2}{(1+u^2)^2}\right]\phi_1=-i18\lambda\left[\frac{8}{\lambda}\right]^{1/2}F\frac{u^5}{(1+u^2)^5}\ , \tag{28}$$

the condition on the appropriate solution to this equation being that its expansion in $u$ starts with the fifth power, since only starting from final states with five particles the threshold amplitudes develop an imaginary part, which in this calculation originates in the imaginary part of $F$. The solution satisfying this condition is

$$\phi_1(\tau)=i\frac{3\lambda}{4}\left[\frac{8}{\lambda}\right]^{1/2}F\frac{u^5}{(1+u^2)^3}\ . \tag{29}$$

Using Eq. (14) one can readily restore from here the response of the field in terms of $z(t)$ with the first quantum correction included,

$$\phi_{0+1}(t)=\frac{z(t)}{1-(\overline{\lambda}/8\overline{m}^2)z(t)^2}\times\left[1-\frac{3\lambda}{4}F\frac{(\lambda/8m^2)^2z(t)^4}{[1-(\lambda/8m^2)z(t)^2]^2}\right]\ , \tag{30}$$

and by expanding in series in $z(t)$ finally arrive at the result in Eq. (2).

The rotation (14) used here may invite the objection that such rotation in the path integral is obstructed by the infinite chains of poles parallel to the real axis of $t$, which may give rise to extra contributions in the quantum effects. However, it can be explicitly shown that this does not happen at least at the one-loop level. Namely, it is a straightforward (but rather cumbersome) exercise to verify that the recursion relations for the sum of graphs for the propagator of the field $\phi$ with emission of $n$ on-shell particles all being at rest are equivalent to the differential equation for the Green's function of the operator (17) and then that the recursion relations for the loop graphs are equivalent to the equation (28) on the $\tau$ axis. Another simple (and in no way rigorous) check is to verify the formula (2) for few first $n$ by direct computation of the graphs. This also turned out to be helpful in checking the relative coefficients and signs in the equations of this paper. The remarkably simple form of the result (2) suggests that there may be a way to calculate further quantum effects. In particular one can notice that the finite term, proportional to the factor $F$ [Eq. (25)], has the form given by the simple scalar vacuum polarization at $q^2=16m^2$. This of course is a consequence of the eigenmode of the operator (17) at $\omega^2=4$, or, equivalently, of the fact that the tree-level threshold amplitudes of the processes $2\to n$ are equal to zero for $n>4$. In terms of the graphs the cancellation of the contributions to the imaginary and the real parts of the thresholds at higher $q^2$ looks quite surprising.

In the present calculation we have avoided approaching the poles of the classical solution $\phi_0$, where the quantum expansion in fact breaks down, since the quantum fluctuations are more singular than the classical solution. However, those are the singularities of the field in the complex plane of $t$ (or equivalently of $z$) which give rise to the factorial growth of the amplitudes. The appearance of the singularity at the imaginary axis follows from the simple fact that on this axis the classical field equation

$$\frac{d^2}{dt^2}\phi=m^2\phi+\lambda\phi^3 \tag{31}$$

corresponds to the free fall in the inverted $\lambda\phi^4$ potential which takes a finite time for a finite staring value $\phi(0)$. It looks at least extremely unnatural that quantum effects would slow down this fall to the extent that the time of the fall would be infinite. For any finite time, however, the singularity of the field there will produce the factorial growth of the amplitudes.

As a simple final remark, it can be mentioned that though the present calculation is done for the case of unbroken symmetry it looks quite straightforward to apply the same technique to the case of the spontaneously broken symmetry.

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[1] A. Ringwald, Nucl. Phys. **B330**, 1 (1990).
[2] O. Espinosa, Nucl. Phys. **B343**, 310 (1990).
[3] For a recent review, see M. P. Mattis, Phys. Rep. **214**, 159 (1992).
[4] J. M. Cornwall, Phys. Lett. B **243**, 271 (1990).
[5] H. Goldberg, Phys. Lett. B **246**, 445 (1990).
[6] For a review see, e.g., J. Zinn-Justin, Phys. Rep. **70**, 109 (1981).
[7] M. B. Voloshin, Nucl. Phys. **B393**, 233 (1992).
[8] E. N. Argyres, R. H. P. Kleiss, and C. G. Papadopoulos, CERN Report No. CERN-TH-6496, 1992 (unpublished).
[9] L. S. Brown, Phys. Rev. D **46**, 4125 (1992).
[10] M. B. Voloshin, Phys. Rev. D **43**, 1726 (1991).
[11] M. B. Voloshin, in Proceedings of the Yale Workshop on Baryon Number Violation at High Energy (unpublished).
[12] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory,* 3rd ed. (Pergamon, New York, 1977).
[13] Wolfram Research, Inc., MATHEMATICA Version 2.1, Wolfram Research, Inc., Champaign, Illinois, 1992.